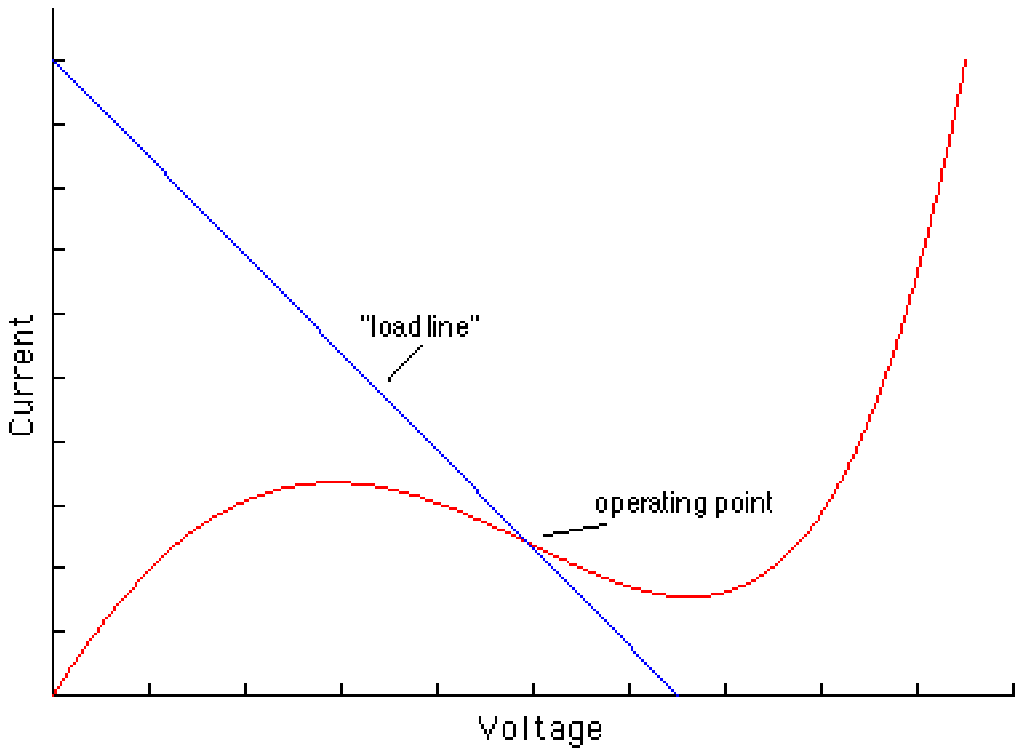
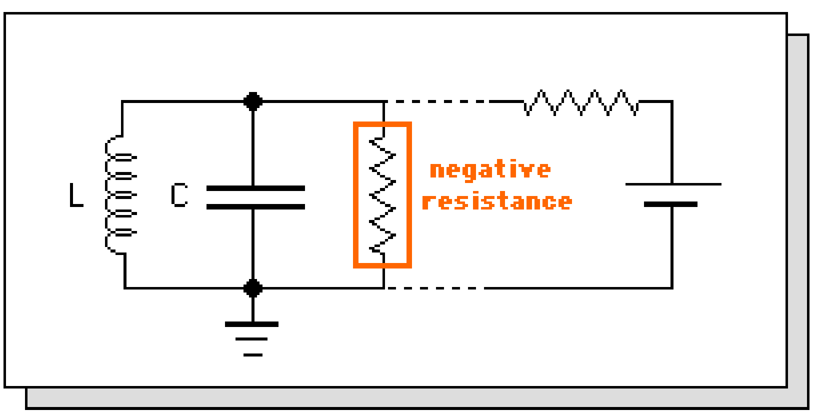
The van der Pol Negative Resistance Oscillator

Van der Pol's analysis[1] of "negative resistance" (*e.g.*, tunnel diode) oscillators prides a valuable framework for treating relative simplicity important features of oscillatory systems.

The characteristic curve of a "negative resistance" device



Consider the following negative resistance oscillatory circuit:



By simple circuit analysis, it is a straightforword proposition to find the following simple circuit equation which is the fundamental van der Pol oscillator equation:

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[1] B. van der Pol, *Radio Rev.* **1**, 704-754, 1920 and B. van der Pol, *Phil. Mag.* **3**, 65, 1927

$$\frac{d^2}{d\,t^2}\, v(t) - \frac{d}{d\,t}\left[\alpha\, v(t) - \bar{\beta}\, v^3(t)\right] + \omega_0^2\, v(t) = 0 \qquad \text{[VdP-1]}$$

where $\omega_0^2 = \left(LC\right)^{-1}$.

If α is small, it is reasonable to take

$$v(t) = \frac{1}{2} V(t)\, \exp\left(-i\,\omega_0\, t\right) + c.c. \qquad \text{[VdP-2]}$$

Then Equation [VdP-1] becomes without approximation

$$\begin{aligned}
&\left\{\frac{1}{2}\left[-\omega_0^2\, V(t) - i\,2\omega_0\, \dot{V}(t) + \ddot{V}(t)\right] \exp\left(-i\,\omega_0\, t\right) + c.c.\right\} \\
&\quad - \left[\alpha - 3\,\bar{\beta}\, v^2(t)\right] \left\{\frac{1}{2}\left[-i\,\omega_0\, V(t) + \dot{V}(t)\right] \exp\left(-i\,\omega_0\, t\right) + c.c.\right\} \\
&\quad + \omega_0^2 \left\{\frac{1}{2} V(t)\, \exp\left(-i\,\omega_0\, t\right) + c.c.\right\} = 0
\end{aligned} \qquad \text{[VdP-3]}$$

If we ignore harmonic generation, Equation [VdP-3] may be approximated as

$$\begin{aligned}
&\left\{\frac{1}{2}\left[-i\,2\,\omega_0\, \dot{V}(t) + \ddot{V}(t)\right] \exp\left(-i\,\omega_0\, t\right) + c.c.\right\} \\
&\quad - \alpha \left\{\frac{1}{2}\left[-i\,\omega_0\, V(t) + \dot{V}(t)\right] \exp\left(-i\,\omega_0\, t\right) + c.c.\right\} \\
&\quad + \frac{3}{8}\,\bar{\beta} \left[\left\{-i\,\omega_0\, |\,V(t)\,|^2\, V(t) + \frac{d}{d\,t}\left[|\,V(t)\,|^2\, V(t)\right]\right\} \exp\left(-i\,\omega_0\, t\right) + c.c.\right] = 0
\end{aligned} \qquad \text{[VdP-4]}$$

If we make the **slow time variation** assumption, this equation reduces to

$$\dot{V}(t) = \frac{1}{2}\left[\alpha - \frac{3}{4}\,\bar{\beta}\, |\,V(t)\,|^2\right] V(t) = \frac{1}{2}\left[\alpha - \beta\, |\,V(t)\,|^2\right] V(t) \qquad \text{[VdP-5]}$$

where $\beta = 3\,\bar{\beta}/4$. This essential Equation [VI-25b] in the lecture set entitled *The Interaction of Radiation and Matter: Semiclassical Theory*. We saw there that the general steady state solution is given by

$$|\,V(t)\,|^2 = \frac{\alpha}{\beta} \qquad \text{[VdP-6]}$$

To study **frequency locking** we suppose that a driving source (to be precise a current source in parallel with the negative resistance) and then the van der Pol equation becomes

$$\frac{d^2}{d\,t^2} v(t) - \frac{d}{d\,t}\left[\alpha\, v(t) - \beta\, v^3(t)\right] + \omega_0^2\, v(t) = \omega^2\, V_0 \sin \omega t$$
$$= \frac{\omega^2 V_0}{2}\left[i \exp\left(-i\,\omega\, t\right) + c.c.\right] \qquad \text{[VdP-7]}$$

In this case, it is reasonable to take

$$v(t) = \frac{1}{2} V(t) \exp\left(-i\,\omega\, t\right) + c.c. \qquad \text{[VdP-8]}$$

If we again ignore harmonic geneation, Equation [VdP-7] becomes

$$\left\{\frac{1}{2}\left[-\omega^2 V(t) - i\,2\,\omega\,\dot{V}(t) + \ddot{V}(t)\right] \exp\left(-i\,\omega t\right) + c.c.\right\}$$
$$-\left[\alpha - 3\,\beta\, v^2(t)\right]\left\{\frac{1}{2}\left[-i\,\omega V(t) + \dot{V}(t)\right] \exp\left(-i\,\omega t\right) + c.c.\right\}$$
$$+\omega_0^2\left\{\frac{1}{2} V(t) \exp\left(-i\,\omega t\right) + c.c.\right\} \qquad \text{[VdP-9]}$$
$$= \frac{\omega^2 V_0}{2}\left[i \exp\left(-i\,\omega\, t\right) + c.c.\right]$$

Again under the **slow time variation** assumption, this equation reduces to

$$\frac{1}{2}\left(\omega_0^2 - \omega^2\right) V(t) - i\,\omega \dot{V}(t) + i\,\omega \frac{1}{2}\left[\alpha - \beta \left| V(t)\right|^2\right] V(t) = i\,\frac{\omega^2 V_0}{2} \qquad \text{[VdP-10]}$$

If we take $V(t) = \left| V(t) \right| \exp\left(-i\,\Psi(t)\right)$, this equation separate into the following pair of equations:

$$\left|\dot{V}(t)\right| = \frac{1}{2}\left[\alpha - \beta \left| V(t)\right|^2\right]\left| V(t)\right| - \frac{\omega V_0}{2} \cos \Psi(t) \qquad \text{[VdP-11a]}$$

$$\dot{\Psi}(t) = \frac{1}{2}\frac{\left(\omega_0^2 - \omega^2\right)}{\omega} + \frac{\omega}{2}\frac{V_0}{\left| V(t)\right|} \sin \Psi(t) = d + l \sin \Psi(t) \qquad \text{[VdP-11b]}$$

where $d \equiv \frac{1}{2}\frac{\left(\omega_0^2 - \omega^2\right)}{\omega} = \left(\omega_0 - \omega\right)$ (the "detuning term") and $l \equiv \frac{\omega}{2}\frac{V_0}{\left| V(t)\right|}$ (the "locking coefficient”). For small V_0 we can decouple the equations and take $|V(t)|^2 = \frac{\alpha}{\beta}$ from Equation [VdP-6] so that $l \equiv \frac{\omega}{2}\frac{V_0}{|V(t)|} = \frac{\omega}{2}\frac{\beta}{\alpha}V_0$. If $|d/l| >> 1$ the relative phase angle changes linearly in time at the rate $\dot{\Psi} \approx d$. As $|d/l|$ decreases toward unity, the “locking term” subtracts from the “detuning term” in one half of a cycle and adds in the other half. At $|d/l| \leq 1$ there are two values of the phase angle that yield the “mode locking” condition $\dot{\Psi} = 0$ -- *viz*.

$$\Psi_{PL} = \begin{cases} -\sin^{-1}(d/l) \\ \pi + \sin^{-1}(d/l) \end{cases} \qquad \text{[VdP-12]}$$

We can test the stability of these solutions by taking $\Psi(t) = \Psi_{PL} + \varepsilon(t)$ and therefore Equation [VdP-11b] becomes

$$\dot{\varepsilon}(t) \approx l\cos(\Psi_{PL})\,\varepsilon(t) \qquad \text{[VdP-13]}$$

and the solutions are stable if

$$l\cos(\Psi_{PL}) < 0 \qquad \text{[VdP-14a]}$$

$$\sqrt{l^2 - d^2} < 0 \qquad \text{[VdP-14b]}$$

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